Exercise 12

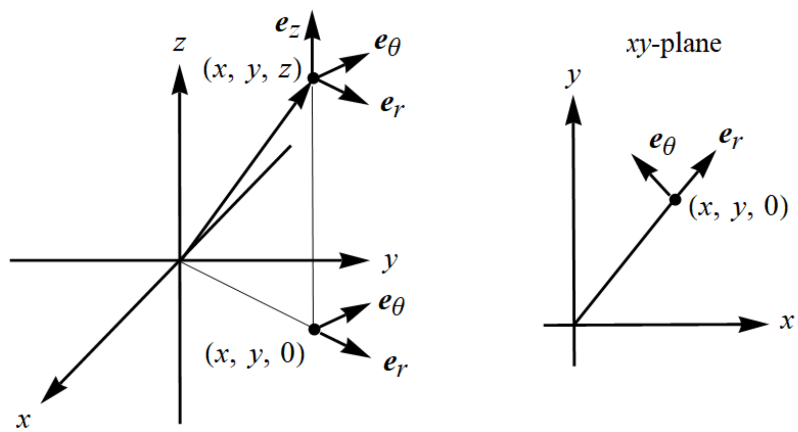
Using cylindrical coordinates and the orthonormal (orthogonal normalized) vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$ (see Figure 1.4.8),

(a) express each of $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$ in terms of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and (x, y, z); and

(b) calculate $\mathbf{e}_\theta \times \mathbf{j}$ both analytically, using part (a), and geometrically.

Solution

The relevant part of Figure 1.4.8 is shown here.



Start by calculating the radial unit vector.

$$\mathbf{e}_r = \frac{\mathbf{r}}{\|\mathbf{r}\|} = \frac{(x, y, 0)}{\sqrt{x^2 + y^2}} = \frac{(x, y, 0)}{r} = \left(\frac{x}{r}, \frac{y}{r}, 0\right) = \frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j}$$

The azimuthal unit vector is perpendicular to the radial unit vector in the xy-plane, which means their dot product is zero.

$$\mathbf{e}_\theta \cdot \frac{(x, y, 0)}{\sqrt{x^2 + y^2}} = 0 \quad \Rightarrow \quad \mathbf{e}_\theta = \frac{(\pm y, \mp x, 0)}{\sqrt{x^2 + y^2}}$$

Since $\mathbf{e}_\theta$ points to the upper left, we choose

$$\mathbf{e}_\theta = \frac{(-y, x, 0)}{\sqrt{x^2 + y^2}} = \frac{(-y, x, 0)}{r} = \left(-\frac{y}{r}, \frac{x}{r}, 0\right) = -\frac{y}{r}\mathbf{i} + \frac{x}{r}\mathbf{j}.$$

Finally,

$$\mathbf{e}_z = \mathbf{k}.$$

Use these formulas to determine the desired cross product.

$$\begin{aligned} \mathbf{e}_\theta \times \mathbf{j} &= \left(-\frac{y}{r}\mathbf{i} + \frac{x}{r}\mathbf{j}\right) \times \mathbf{j} \\ &= -\frac{y}{r}(\mathbf{i} \times \mathbf{j}) + \frac{x}{r}(\mathbf{j} \times \mathbf{j}) \\ &= -\frac{y}{r}(\mathbf{k}) \\ &= -\frac{y}{r}\mathbf{k} \end{aligned}$$